AP® Calculus BC Exam

SECTION I: Multiple-Choice Questions

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

At a Glance

Total Time
1 hour and 45 minutes
Number of Questions
45
Percent of Total Grade
50%
Writing Instrument
Pencil required

Instructions

Section I of this examination contains 45 multiple-choice questions. Fill in only the ovals for numbers 1 through 45 on your answer sheet.

CALCULATORS MAY NOT BE USED IN THIS PART OF THE EXAMINATION.

Indicate all of your answers to the multiple-choice questions on the answer sheet. No credit will be given for anything written in this exam booklet, but you may use the booklet for notes or scratch work. After you have decided which of the suggested answers is best, completely fill in the corresponding oval on the answer sheet. Give only one answer to each question. If you change an answer, be sure that the previous mark is erased completely. Here is a sample question and answer.

Sample Question	Sample Answer
Chicago is a (A) state (B) city (C) country (D) continent (E) village	(A) ● (C) (D) (E)

Use your time effectively, working as quickly as you can without losing accuracy. Do not spend too much time on any one question. Go on to other questions and come back to the ones you have not answered if you have time. It is not expected that everyone will know the answers to all the multiple-choice questions.

About Guessing

Many candidates wonder whether or not to guess the answers to questions about which they are not certain. Multiple choice scores are based on the number of questions answered correctly. Points are not deducted for incorrect answers, and no points are awarded for unanswered questions. Because points are not deducted for incorrect answers, you are encouraged to answer all multiple-choice questions. On any questions you do not know the answer to, you should eliminate as many choices as you can, and then select the best answer among the remaining choices.

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CALCULUS BC

SECTION I, Part A

Time—55 Minutes

Number of questions—28

A CALCULATOR MAY NOT BE USED ON THIS PART OF THE EXAMINATION

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding oval on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test: Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. $\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 5x + 6} =$

(A) 0
(B) 1
(C) 3
(D) 5
(E) The limit does not exist.

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2. $\lim_{x \to 0} \frac{\sec x}{\csc x} =$

(A) 0
(B) 1
(C) 2π
(D) ∞
(E) The limit does not exist.

GO ON TO THE NEXT PAGE.

3. $\dfrac{d}{dx}\left(\dfrac{2x+3}{(x-4)^2}\right) =$

(A) $\dfrac{x-7}{(x-4)^3}$

(B) $\dfrac{-2x-14}{(x-4)^3}$

(C) $\dfrac{2x+14}{(x-4)^3}$

(D) $\dfrac{-2x-14}{(x-4)^4}$

(E) $\dfrac{x+14}{(x-4)^4}$

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4. $\lim\limits_{x\to 0}\left(\dfrac{2\sin 3x}{3\sin 2x}\right) =$

(A) –1
(B) 0
(C) 1
(D) ∞
(E) The limit does not exist.

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5. What type(s) of discontinuity/ies does the function, $f(x) = \dfrac{x^2-7x-18}{x^2-12x+27}$, have?

(A) jump
(B) point
(C) essential
(D) jump and removable
(E) essential and removable

GO ON TO THE NEXT PAGE.

6. Find the fourth derivative of $f(x) = \frac{x^4 - 4x^3 + 3x^2 - 4x}{x^3}$.

(A) $\frac{x^3 - 3x + 8}{x^3}$

(B) $\frac{6x - 24}{x^4}$

(C) $\frac{-18x + 96}{x^5}$

(D) $\frac{x^3 - 12x^2 + 3x - 4}{x^2}$

(E) $\frac{7x - 480}{x^6}$

---

7. $\frac{d}{dx}\left(\frac{\sec 5x}{5}\right) =$

(A) $\tan 5x$
(B) $\sec 5x$
(C) $\csc 5x$
(D) $\sec 5x \tan 5x$
(E) $5\sec 5x \tan 5x$

---

8. What is the volume of the solid formed by rotating the region between the curves $y = 6x^2 - x$ and $y = x^2 - 6x$ about the y-axis?

(A) $\frac{70\pi}{12}$

(B) 5π

(C) $\frac{3\pi}{2}$

(D) π

(E) $\frac{5\pi}{6}$

GO ON TO THE NEXT PAGE.

9. $\int x^3 \ln 2x \, dx =$

(A) $\frac{x^4}{16}(4\ln 2x - 1) + C$

(B) $\frac{x^4}{4}(\ln 2x + 1) + C$

(C) $x^4(\ln 2x - 1) + C$

(D) $\frac{x^4}{4}(\ln 2x - 1) + C$

(E) $\frac{x^4}{16}(4\ln 2x + 1) + C$

---

10. $\int \frac{x+10}{2x^2 - 5x - 3} \, dx =$

(A) $3\ln|x+3| - \frac{1}{2}\ln|2x-1| + C$

(B) $\ln|x+3| - 3\ln|2x-1| + C$

(C) $\ln|x+3| - \frac{3}{2}\ln|2x-1| + C$

(D) $3\ln|x+3| + \frac{3}{2}\ln|2x-1| + C$

(E) $-\ln|x+3| + \frac{3}{2}\ln|2x-1| + C$

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11.

x	-2	-1	0	1	2
$f''(x)$	4	1	0	-2	3

The polynomial function f has selected values of its second derivative, f'', given in the table above. Which of the following could be false?

A) The graph of f changes concavity in the interval $(-2, 1)$.
(B) There is a point of inflection on the graph of f at $x = 0$.
(C) The graph of f has a point of inflection at $x = 1.5$.
(D) The graph of f is concave down at $x = 1$.
(E) The graph of f changes concavity in the interval $(1, 2)$.

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12. If $y = \sqrt[3]{\dfrac{(x+2)^2(x-4)^3}{x^3-1}}$, $\dfrac{dy}{dx} =$

(A) $y\left(\dfrac{2}{3(x+2)} + \dfrac{1}{x-4} - \dfrac{x^2}{x^3-1}\right)$

(B) $y\left(\dfrac{2}{x+2} + \dfrac{3}{x-4} - \dfrac{3x^2}{x^3-1}\right)$

(C) $y\left(\dfrac{2}{3(x+2)} - \dfrac{1}{x-4} - \dfrac{x^2}{x^3-1}\right)$

(D) $y\left(\dfrac{2}{3(x+2)} - \dfrac{1}{x-4} + \dfrac{x^2}{x^3-1}\right)$

(E) $y\left(\dfrac{2}{x+2} - \dfrac{3}{x-4} + \dfrac{3x^2}{x^3-1}\right)$

GO ON TO THE NEXT PAGE.

13. If $y = x^3 + 3x^2 + 5$, then $\dfrac{dy}{dx} =$

(A) $3x^3 + 6x^2 + 5$

(B) $3x^2 + 6x$

(C) $\dfrac{x^4}{4} + x^3 + 5x$

(D) $15x^5$

(E) $3x^2 + 6x + 5$

---

14. If $f(x) = \dfrac{x^3 + 2x - 1}{x^2 + x}$, then $f'(x) =$

(A) $\dfrac{(x^2 + x)(3x^2 + 2) - (x^3 + 2x - 1)(2x + 1)}{(x^2 + x)}$

(B) $\dfrac{(x^2 + x)(3x + 2) - (x^3 + 2x - 1)(2x + 1)}{(x^2 + x)^2}$

(C) $\dfrac{(x^2 + x)(3x^2 + 2) - (x^3 + 2x - 1)(2x + 1)}{(x^2 + x)^2}$

(D) $\dfrac{(x^2 + x)(3x^2 + 2) - 2x(x^3 + 2x - 1)}{(x^2 + x)^2}$

(E) $\dfrac{(x^2 + x)(3x^2 + 2) + (x^3 + 2x - 1)(2x + 1)}{(x^2 + x)^2}$

GO ON TO THE NEXT PAGE.

15. If $f(x) = 2\sin(\cos x)$, then $f'(x) =$

(A) $-2\sin x \bullet \cos(\cos x)$
(B) $-2\cos x \bullet \sin(\sin x)$
(C) $2\sin x \bullet \cos x$
(D) $-2\sin x \bullet \cos x$
(E) $2\sin x \bullet \cos(\cos x)$

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16. Find $\frac{dy}{dx}$ if $x^3 + 3x^2y + 3xy^2 + y^3 = 27$.

(A) 27

(B) $\frac{3x^2+3y^2}{6x}$

(C) 1

(D) $\frac{-3x^2-3y^2}{6x}$

(E) –1

---

17. Water is filling a conical cup at a rate of $\frac{2}{3}\pi$ in^3/sec. If the cup has a height of 18 in and a radius of 6 in, how fast is the water level rising when the water is 6 in deep?

(A) $\frac{1}{6}$ in/s

(B) 6 in/s

(C) $\frac{1}{4}$ in/s

(D) $\frac{8}{3}$ in/s

(E) $\frac{1}{12}$ in/s

GO ON TO THE NEXT PAGE.

18. Find the derivative of $\log_8\left(x^2+2\right)^3$

(A) $\dfrac{6x}{\left(x^2+2\right)^3 \ln 8}$

(B) $\dfrac{6x}{\left(x^2+2\right)\ln 8}$

(C) $\dfrac{6x\left(x^2+2\right)^3}{\ln 8}$

(D) $\dfrac{6}{\left(x^2+2\right)\ln 8}$

(E) $\dfrac{6x \ln 8}{\left(x^2+2\right)}$

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19. What curve is represented by $x = \cos^2 t$ and $y = \sin^2 t$?

(A) $y = x + 1$
(B) $y = -x + 1$
(C) $y = -x - 1$
(D) $y = x - 1$
(E) $y = x$

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20. Given the position function $x(t) = t^3 - 18t^2 - 84t + 11$ for $t \geq 0$, for what values of t is speed increasing?

(A) $0 \leq t < 14$
(B) $0 \leq t < 6$ and $t > 14$
(C) $0 \leq t < 6$
(D) $t > 6$
(E) $6 < t < 14$

GO ON TO THE NEXT PAGE.

21. Find the derivative of $f\left(x\right)=e^{\sin^2 x}$

(A) $2e^{\sin^2 x}\sin x\cos x$

(B) $e^{\sin^2 x}$

(C) $e^{\sin^2 x}\sin^2 x$

(D) $2e^{\sin^2 x}\sin x$

(E) $e^{\sin^2 x}\sin x\cos x$

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22. Given:

x	0	2	6	7	9	12	16
$f(x)$	1	3	7	5	3	6	9

Use a left-hand Riemann sum with the six subintervals indicated by the data to approximate $\int_0^{16} f\left(x\right)dx$.

(A) 63
(B) 99
(C) 40
(D) 64
(E) 100

---

23. $\int\frac{18x^2+9}{3x^3+x}dx=$

(A) $2\ln\left|x^2+1\right|+C$

(B) $\frac{1}{2}\ln\left|3x^3+x\right|+C$

(C) $\ln\left|3x^3+x\right|+C$

(D) $\ln\left|x^2+1\right|+C$

(E) $2\ln\left|3x^3+x\right|+C$

GO ON TO THE NEXT PAGE.

24. $\int e^x \sin x \, dx =$

(A) $\dfrac{e^x \sin x - e^x \cos x}{2} + C$

(B) $\dfrac{e^x \sin x + e^x \cos x}{2} + C$

(C) $e^x \sin x - e^x \cos x + C$

(D) $e^x \cos x - e^x \sin x + C$

(E) $\dfrac{e^x \cos x - e^x \sin x}{2} + C$

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25. Which of the following integrals converges?

I. $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$

II. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{1-x^2}}$

III. $\displaystyle\int_1^\infty \frac{dx}{x}$

(A) I
(B) II
(C) III
(D) I & II
(E) I, II, & III

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26. Find the area of the region in the plane enclosed by $r = 5 + 2 \cos \theta$.

(A) 23π
(B) 24π
(C) 25π
(D) 26π
(E) 27π

GO ON TO THE NEXT PAGE.

27. If $\frac{dy}{dx} = \frac{\cos x}{y^2}$ and $y = -1$ when $x = 0$, then the equation for the curve is

(A) $y^3 = 3\sin x - 1$
(B) $y^3 = 3\cos x - 1$
(C) $y = \sin x - 1$
(D) $y = 3\sin x + 1$
(E) $y^3 = \sin x$

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28. Which of the following series converges?

I. $\sum_{n=1}^{\infty} 2\left(\frac{1}{3}\right)^{n-1}$

II. $\sum_{n=1}^{\infty} \frac{n}{3^n}$

III. $\sum_{n=1}^{\infty} \frac{6n^2}{n^3+1}$

(A) I
(B) II
(C) III
(D) I & II
(E) I & III

END OF PART A, SECTION I

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY.

DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

CALCULUS BC

SECTION I, Part B

Time—50 Minutes

Number of questions—17

A GRAPHING CALCULATOR IS REQUIRED FOR SOME QUESTIONS ON THIS PART OF THE EXAMINATION

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding oval on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test:

1. The **exact** numerical value of the correct answer does not always appear among the choices given. When this happens, select from among the choices the number that best approximates the exact numerical value.

2. Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

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29. A spherical balloon is losing air at a rate of -24π in^3/sec. How fast is the balloon surface area of the balloon shrinking when the radius of the balloon is 2 in?

(A) -20π in^2/sec
(B) -24π in^2/sec
(C) -48π in^2/sec
(D) -12π in^2/sec
(E) -60π in^2/sec

GO ON TO THE NEXT PAGE.

30. Find $\lim_{x \to 0} \frac{(3+2x)^{\frac{3}{2}} - 7x}{2x^2}$.

(A) $\frac{\sqrt{3}}{4}$

(B) $\frac{4\sqrt{3}}{3}$

(C) $\frac{3}{4}$

(D) $\frac{1}{4}$

(E) $\frac{1}{3}$

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31. When are the horizontal and vertical components of the velocity of a curve whose motion is given by $x = \frac{5}{2}t^2 + 6$ and $y = 2t^3 - t^2 + t$ equal?

(A) $t = -1$

(B) $t = 2$

(C) $t = \frac{1}{6}$

(D) $t = 6$

(E) $t = -2$

GO ON TO THE NEXT PAGE.

32. Find the derivative of $x^2 + 3x^2y + 3xy^2 + y^3 = 2$ at (3,2).

(A) 7

(B) $\frac{19}{25}$

(C) $\frac{75}{54}$

(D) $-\frac{18}{25}$

(E) $-\frac{25}{18}$

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33. A kid on a bike is riding home in the woods on a straight path that is 50 m from the nearest point on the road. His home is 1500 m from the nearest point on the road. If the kid rides at 3 m/s in the woods and 5 m/s on the road, how far from his house should the kid cross to the road to get home in the shortest time?

(A) 37.5 m
(B) 1500 m
(C) 300.5 m
(D) 1200 m
(E) 1462.5 m

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34. A shoe company determined that its profit equation (in millions of dollars) is given by $P = 2x^3 - 105x^2 + 1500x - 1200$, where x is the number of thousands of pairs of shoes sold and $0 \le x \le 50$. Optimize the manufacturer's profit.

(A) \$5.3 billion
(B) \$61.3 billion
(C) \$1.925 billion
(D) \$1.2 billion
(E) \$65 billion

GO ON TO THE NEXT PAGE.

35. If the function $f(x) = x^4$ has an average value of 5 on the closed interval [0, k], then $k =$

(A) 5
(B) $\sqrt{5}$
(C) 1
(D) $\sqrt{3}$
(E) $\sqrt{2}$

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36. $\int 3x(7^{3x^2+2})dx =$

(A) $\dfrac{49 \bullet 343^{x^2}}{\ln 49} + C$

(B) $\dfrac{343^{x^2}}{2\ln 7} + C$

(C) $\dfrac{49^{x^2}}{\ln 49} + C$

(D) $\dfrac{49 \bullet 343^{x^2}}{\ln 7} + C$

(E) $\dfrac{7^{3x^2+2}}{\ln 7} + C$

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37. A rectangle with one side on the x-axis has its upper vertices on the graph of $y = \cos x + 1$. What is the minimum area between the rectangle and the graph of $y = \cos x + 1$ on the interval $(-\pi \le x \le \pi)$?

(A) 6.283
(B) 2.988
(C) 1.307
(D) 3.296
(E) 5.022

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38. $\int \frac{dx}{x^2 + 6x + 10} =$

(A) $\cot^{-1}(x + 3) + C$
(B) $\sin^{-1}(x + 3) + C$
(C) $\sec^{-1}(x + 3) + C$
(D) $\tan^{-1}(x + 3) + C$
(E) $\cos^{-1}(x + 3) + C$

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39. If the path of the particle is given by $x(t) = 2t^2 - 3t + 1$, how far does the particle travel between $t = 0$ and $t = 4$?

(A) 20

(B) 21

(C) $\frac{169}{8}$

(D) 22

(E) $\frac{89}{4}$

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40. Given the following values for x and $f(x)$, what is the area under $f(x)$. Use a left-hand Riemann sum to approximate.

x	0	1	3	7	8	10	13	15
$f(x)$	2	6	3	4	8	9	12	13

(A) 46
(B) 57
(C) 116
(D) 207
(E) 253

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41. Find the length of the curve given by $y=\frac{1}{3}\left(x^2-2\right)^{\frac{3}{2}}$ from $x = 0$ to $x = 4$.

(A) 12
(B) 16
(C) $\frac{64}{3}-4$
(D) $\frac{64}{3}$
(E) $\frac{64}{3}+4$

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42. Find $\frac{d}{dx}\int_1^x \left(t-t^4\right)dt$.

(A) $x - x^4 + 1$
(B) $x^4 - x + 1$
(C) $x^4 - x$
(D) $x - x^4 - 1$
(E) $x - x^4$

---

43. A cylindrical pool is filling at a rate of $96\pi\ \frac{\text{ft}^3}{\text{hr}}$. If the radius of the pool is 4 feet, how fast is the height changing?

(A) 3 ft/hr
(B) 4 ft/hr
(C) 5 ft/hr
(D) 6 ft/hr
(E) 7 ft/hr

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44. A child jumps on a trampoline and rises at a rate of 10 ft/min. The child's mother is watching from the patio 40 ft away. How fast, in rad/sec, is the angle of elevation between the trampoline and the mother's line of sight of her child increasing when the child is 30 ft in the air?

(A) $\frac{1}{70}$ rad/sec

(B) $\frac{1}{375}$ rad/sec

(C) $\frac{1}{13}$ rad/sec

(D) $\frac{1}{180}$ rad/sec

(E) $\frac{1}{3}$ rad/sec

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45. Find the derivative of $y = \frac{\ln\left(2x^3\right)}{e^{2x}}$.

(A) $\frac{3+2\ln\left(2x^3\right)}{e^{2x}}$

(B) $\frac{3-2\ln\left(2x^3\right)}{e^{2x}}$

(C) $\frac{3}{xe^{2x}} - \frac{2\ln\left(2x^3\right)}{e^{2x}}$

(D) $\frac{3}{xe^{2x}} + \frac{2\ln\left(2x^3\right)}{e^{2x}}$

(E) $\frac{3+\ln\left(4x^9\right)}{e^{2x}}$

STOP

END OF PART B, SECTION I

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART B ONLY.

DO NOT GO ON TO SECTION II UNTIL YOU ARE TOLD TO DO SO.

SECTION II
GENERAL INSTRUCTIONS

You may wish to look over the problems before starting to work on them, since it is not expected that everyone will be able to complete all parts of all problems. All problems are given equal weight, but the parts of a particular problem are not necessarily given equal weight.

A GRAPHING CALCULATOR IS REQUIRED FOR SOME PROBLEMS OR PARTS OF PROBLEMS ON THIS SECTION OF THE EXAMINATION.

- You should write all work for each part of each problem in the space provided for that part in the booklet. Be sure to write clearly and legibly. If you make an error, you may save time by crossing it out rather than trying to erase it. Erased or crossed-out work will not be graded.
- Show all your work. You will be graded on the correctness and completeness of your methods as well as your answers. Correct answers without supporting work may not receive credit.
- Justifications require that you give mathematical (noncalculator) reasons and that you clearly identify functions, graphs, tables, or other objects you use.
- You are permitted to use your calculator to solve an equation, find the derivative of a function at a point, or calculate the value of a definite integral. However, you must clearly indicate the setup of your problem, namely the equation, function, or integral you are using. If you use other built-in features or programs, you must show the mathematical steps necessary to produce your results.
- Your work must be expressed in standard mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\,dx$ may not be written as fnInt (X^2, X, 1, 5).
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.
- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

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SECTION II, PART A
Time—30 minutes
Number of problems—2

A graphing calculator is required for some problems or parts of problems.

During the timed portion for Part A, you may work only on the problems in Part A.

On Part A, you are permitted to use your calculator to solve an equation, find the derivative of a function at a point, or calculate the value of a definite integral. However, you must clearly indicate the setup of your problem, namely the equation, function, or integral you are using. If you use other built-in features or programs, you must show the mathematical steps necessary to produce your results.

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1. Let y be the function satisfying $f'(x) = -x(1 + f(x))$ and $f(0) = 5$.

   (a) Use Euler's method to approximate $f(1)$ with a step size of 0.25.

   (b) Find an exact solution for $f(x)$ when $x = 1$.

   (c) Evaluate $\int_0^\infty -x\left(1 + f\left(x\right)\right)dx$.

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2. Let R be the region enclosed by the graphs of $y = x^2 - x - 6$ and $y = x^3 - 2x^2 - 5x + 6$ and the lines $x = -2$ and $x = 2$.

   (a) Find the area of R.
   (b) The horizontal line $y = 0$ splits R into two parts. Find the area of the part of R above the horizontal line.
   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an equilateral triangle. Find the volume of this solid.
   (d) What is the volume of the solid generated by the region R when it is revolved about the line $x = -3$.

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SECTION II, PART B
Time—1 hour
Number of problems—4

No calculator is allowed for these problems.

During the timed portion for Part B, you may continue to work on the problems in Part A without the use of any calculator.

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3. The derivative of a function f is $f'(x) = (2x+6)e^{-x}$ and $f(2) = 15$.

   (a) The function has a critical point at $x = -3$. Is there a relative maximum, minimum, or neither at this point on f? Justify your response.
   (b) On what interval, if any, is the graph of f both increasing and concave down? Explain your reasoning.
   (c) Find the value of $f(5)$.

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4. Consider the equation $x^3 - 2x^2y + 3xy^2 - 4y^3 = 10$.

   (a) Write an equation for the slope of the curve at any point.

   (b) Find the equation of the normal line to the curve at the point $x = 1$.

   (c) Find $\frac{d^2y}{dx^2}$ at $x = 1$.

---

5. Given that $f(x) = \sin x$:

   (a) Find the 6^{th} degree Maclaurin series.
   (b) Use the polynomial to estimate sin 0.2.
   (c) Estimate the remainder of the approximation.

GO ON TO THE NEXT PAGE.

6. Two particles travel in the xy-plane for time $t \geq 0$. The position of particle A is given by $x = 2t - 3$ and $y = (2t + 1)^2$ and the position of particle B is given by $x = t - 1$ and $y = t + 23$.

(a) Find the velocity vector for each particle at $t = 3$.
(b) Set up, but do not evaluate, an integral expression for the distance traveled by particle A from $t = 3$ to $t = 5$.
(c) At what time do the two particles collide? Justify your answer.

STOP

END OF EXAM

ANSWER KEY

Section I

1. D
2. A
3. B
4. C
5. E
6. E
7. D
8. E
9. A
10. E
11. C
12. A
13. B
14. C
15. A
16. E
17. A
18. B
19. B
20. B
21. A
22. D
23. E
24. A
25. A
26. E
27. A
28. D
29. B
30. A
31. C
32. D
33. E
34. B
35. B
36. A
37. B
38. D
39. E
40. E
41. C
42. E
43. D
44. B
45. C

EXPLANATIONS

Section I

1. **D** $\lim_{x\to 3}\dfrac{x^2-x-6}{x^2-5x+6}=\lim_{x\to 3}\dfrac{(x+2)(x-3)}{(x-2)(x-3)}=\lim_{x\to 3}\dfrac{x+2}{x-2}=\dfrac{5}{1}=5.$

2. **A** $\lim_{x\to 0}\dfrac{\sec x}{\csc x}=\lim_{x\to 0}\dfrac{\sin x}{\cos x}=\lim_{x\to 0}\tan x=0.$

3. **B** $\dfrac{d}{dx}\left(\dfrac{2x+3}{(x-4)^2}\right)=\dfrac{2(x-4)^2-2(2x+3)(x-4)}{(x-4)^4}=\dfrac{2(x-4)-2(2x+3)}{(x-4)^3}=\dfrac{-2x-14}{(x-4)^3}$

4. **C** $\lim_{x\to 0}\left(\dfrac{2\sin 3x}{3\sin 2x}\right)=\left(\dfrac{2}{3}\right)\lim_{x\to 0}\left(\dfrac{\sin 3x}{\sin 2x}\right)=\dfrac{2}{3}\left(\dfrac{3}{2}\right)=1.$

5. **E** Jump discontinuities exist when the left and right hand limits of the function are not equal at a certain value. A point discontinuity exists when the limit of the function at a certain value does not equal the function at that value. An essential discontinuity is a vertical asymptote. A removable discontinuity occurs where a canceled factor in a rational expression existed. For this function, $f(x) = \dfrac{(x+2)(x-9)}{(x-9)(x-3)}$, since the factor $(x - 9)$ can be factored out, there will be a removable discontinuity at $x = 9$. Also, since the function does not exist at $x = 3$, there is a vertical asymptote there, so there is both a removable and an essential discontinuity on this function's graph.

6. **E** First, the function can simplify to $f(x) = \dfrac{x^3-12x^2+3x-4}{x^2}$. Use the quotient rule and be prepared to simplify to make the multiple derivatives easier to evaluate. The first derivative is $f'(x) = \dfrac{x^2(3x^2-24x+3)-2x(x^3-4x^2+3x-4)}{x^4}=\dfrac{x^3-3x+8}{x^3}$. The second derivative is $f''(x) =$ $f''(x)=\dfrac{(x^3)(3x^2-3)-(3x^2)(x^3-3x+8)}{x^6}=\dfrac{6x-24}{x^4}$. The third derivative is $f'''(x) = \dfrac{6x^4-4x^3(6x-24)}{x^8}=\dfrac{-18x+96}{x^5}$. Finally, the fourth derivative is $f^{(4)}(x) = \dfrac{-18x^5-5x^4(-18x+96)}{x^{10}}=\dfrac{72x-480}{x^6}$.

7. **D** Use u-substitution and remember your derivatives of trig functions.

$$\frac{d}{dx}\left(\frac{\sec 5x}{5}\right) = \frac{1}{5}(5)(\sec 5x\tan 5x) = \sec 5x\tan 5x.$$

8. **E** First, determine where the two curves intersect. Set them equal to each other and solve for x. Thus, $x = 0$ and $x = -1$ at the points of intersection that bind the area. (Determine which curve is "on top" in order to solve properly.) Next, integrate using cylindrical shells from $x = 0$ to $x = -1$: $2\pi\int_0^1 x(6x^2 - x - (x^2 - 6x))\,dx = \dfrac{70\pi}{12}$. The washer method would have worked, but you would have had to convert the equations into "$x =$" form. Recall, it is generally best to use the cylindrical shells method when the axis of rotation is in "$x =$" form and the equations in "$y =$" form. When everything is in "$y =$" form the washer method is generally better.

9. **A** Since u-substitution will not work, you must integrate by parts. Recall, the formula is $\int u\,dv = uv - \int v\,du$. For this problem, $u = \ln 2x$, $du = \frac{dx}{x}$, $dv = x^3\,dx$, and $v = \frac{x^4}{4}$. (If you are unsure of the derivative of ln $2x$, remember the product rule of logarithms and then differentiate.) Now, the integral is $\int x^5 \ln 2x\,dx = \frac{x^4}{4}\ln 2x - \frac{1}{4}\int x^3\,dx = \frac{x^4}{16}(4\ln(2x) - 1) + C$.

10. **E** Use partial fractions to solve this problem. When this is done, the fraction in the integrand becomes $\frac{x+10}{2x^2-5x-3} = \frac{-1}{x+3} + \frac{3}{2x-1}$. If those two fractions are used as integrands, via u-substitution, the solution is $-\ln|x + 3| + \frac{3}{2}\ln|2x - 1| + C$.

11. **C** By the second derivative test, when the second derivative equals zero, there is a point of inflection at that point. A point of inflection signifies a change in the concavity of the graph of the original function. In addition, if the second derivative is negative over an interval, the graph of the original function is concave down over that interval. If the second derivative is positive, the graph is concave up.

12. **A** Rather than deal with the chain, product, and quotient rules, use logarithmic differentiation:

$$\ln y = \frac{1}{3}(2\ln(x+2) + 3\ln(x-4) - \ln(x^3-1))$$
$$= \frac{1}{y}\frac{dy}{dx} = \frac{2}{3(x+2)} + \frac{1}{x-4} - \frac{x^2}{x^3-1}$$
$$= \frac{dy}{dx} = y\left(\frac{2}{3(x+2)} + \frac{1}{x-4} - \frac{x^2}{x^3-1}\right)$$

13. **B** Using the Power and Addition Rules, take the derivative of $f(x)$ and you get B. Be careful, C is the integral.

14. **C** Using the Quotient Rule, $\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, take the derivative of $f(x)$ and you get C.

15. **A** Using the Chain Rule, $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx}$, and $d(\sin x) = \cos x$ and $d(\cos x) = -\sin x$, the answer is A.

16. **E** Use implicit differentiation to find $\frac{dy}{dx}$:

$$3x^2 + 6xy + 3x^2\frac{dy}{dx} + 3y^2 + 6xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

Isolate terms containing $\frac{dy}{dx}$ by moving all terms that don't contain $\frac{dy}{dx}$ to the right side of the equals sign:

$$3x^2\frac{dy}{dx} + 6xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -3x^2 - 6xy - 3y^2$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}(3x^2 + 6xy + 3y^2) = -3x^2 - 6xy - 3y^2$$

Isolate $\frac{dy}{dx}$ and simplify:

$$\frac{dy}{dx} = \frac{-3x^2 - 6xy - 3y^2}{3x^2 + 6xy + 3y^2} = \frac{-(3x^2 + 6xy + 3y^2)}{3x^2 + 6xy + 3y^2} = -1$$

17. **A** The volume of a cone is found from the formula $V = \frac{1}{3}\pi r^2 h$. Because we are told the radius of the cup is 6 in and the height is 18 in, then at any level, the height of the water will be three times the radius, thus, $h = 3r$. Using this relationship, r can be replaced with $\frac{h}{3}$ in the formula for volume, so $V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{\pi h^3}{27}$. Then, to determine the rate that the water level is rising, we must differentiate both sides with respect to t: $\frac{dV}{dt} = \frac{\pi h^2}{9}\frac{dh}{dt}$. Insert the rate of the cup filling $\left(\frac{dV}{dt} = \frac{2}{3}\pi \text{ in}^3/\text{sec}\right)$ and the water level at the instant of interest, 6 in, and solve for $\frac{dh}{dt}$. Thus, $\frac{dh}{dt} = \frac{1}{6}$ in/s.

18. **B** When $y = \log_a u$, $\frac{dy}{dx} = \frac{1}{u \ln a}\frac{du}{dx}$. For this problem, $u = (x^2 + 2)^3$ and $\frac{du}{dx} = 6x(x^2 + 2)^2$ (from the Chain Rule). Therefore, $f'(x) = \frac{6x\left(x^2+2\right)^2}{\left(x^2+2\right)^3 \ln 8} = \frac{6x}{\left(x^2+2\right)\ln 8}$.

19. **B** Since you are relating a sine function and a cosine function, look for a trig identity that easily relates those. In this case, $\cos^2 t + \sin^2 t = 1$. When you replace $\cos^2 t$ and $\sin^2 t$ with x and y, respectively, the equation becomes $x + y = 1$. Solve for y to get $y = -x + 1$.

20. **B** Recall that the velocity function is the first derivative of a position function with respect to time and the acceleration function is the second derivative of a position function with respect to time. Further, when the velocity and acceleration of a particle have the same sign, the speed is increasing. Thus, to solve this problem, first the position function must be differentiated with respect to time: $v(t) = 3t^2 - 36t - 84$. This function is set equal to zero and the critical values, or the times when velocity is equal to 0, are found: $t = -2$ and $t = 14$. As the problem states, we are only interested in $t \geq 0$, so we can ignore $t = -2$. Next, find the sign of the velocity over the time ranges of interest: $0 \leq t < 14$ results in $v(t) < 0$, and $t > 14$ results in $v(t) > 0$. Now to determine when the speed is increasing, differentiate the velocity with respect to time to get the acceleration function: $a(t) = 6t - 36$. Determine when the acceleration is 0 ($t = 6$) and then find the sign of the acceleration around that time: $0 \leq t < 6$ has $a(t) < 0$ and $t > 6$ has $a(t) > 0$. The times when both the velocity and the acceleration have the same sign is when $0 \leq t < 6$ and when $t > 14$.

21. **A** When $y = e^u$, $\frac{dy}{dx} = e^u\frac{du}{dx}$. For this problem, $u = \sin^2 x$ and $\frac{du}{dx} = 2 \sin x \cos x$ (from the Chain Rule). Therefore, $f'(x) = 2e^{\sin^2 x} \sin x \cos x$.

22. **D** The formula for the area under a curve using a left-hand Reimann sum is: $A = \left(\frac{b-a}{n}\right)(y_0 + y_1 + y_1 + y_3 + \ldots + y_n)$, where a and b are the x-values that bound the area and n is the number of rectangles. Since we do not have evenly spaced subintervals, we must find the width of each subinterval individually and multiply it by the left-endpoint y-value, so $A = 1(2) + 3(4) + 7(1) + 5(2) + 3(3) + 6(4) = 64$.

23. **E** Recall, $\int \frac{du}{u} = \ln|u| + C$, so using u-substitution, $u = 3x^3 + x$ and $du = (9x^2 + 1)$. Then, $2\int \frac{du}{u} = 2\ln|u| + C = 2\ln|3x^3 + x| + C$.

24. A This is a complicated integral and integration by parts is the best way to go for solving it. Recall, the formula for integration by parts is $\int u dv = uv - \int v du$. For this problem, let $u = e^x$ and $dv = \sin x\, dx$. From these, $du = e^x\, dx$ and $v = -\cos x$. When you input these using the integration by parts formula you get: $\int e^x \sin x\, dx = -e^x \cos x + \int \cos x e^x\, dx$. Since we cannot readily integrate $\int \cos x e^x\, dx$, we must use integration by parts again. In this case, $u = e^x$ and $dv = \cos x\, dx$. From these, $du = e^x\, dx$ and $v = \sin x$. Insert these in to the equation in place of $\int \cos x e^x\, dx$ and you get: $\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$. We are back to where we started. However, we are in a good position now, because we can add $\int e^x \sin x\, dx$ to

both sides and we end up with $2\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x$. In order to solve for $\int e^x \sin x\, dx$, just divide both sides by 2: $\int e^x \sin x\, dx = \dfrac{e^x \sin x - e^x \cos x}{2} + C$.

25. **A** All the options are improper integrals and they will converge if their limits as they approach infinity exist. So, check them one at a time and POE. I. $\lim\limits_{a\to\infty}\int_0^\infty \dfrac{dx}{1+x^2} = \lim\limits_{a\to\infty}\tan^{-1}a - \tan^{-1}0 = \dfrac{\pi}{2}$, so I. converges. II. $\lim\limits_{a\to\infty}\int_0^\infty \dfrac{dx}{\sqrt{1-x^2}} = \lim\limits_{a\to\infty}\sin^{-1}a - \sin^{-1}0 =$ *undefined*, so II. diverges. III. $\lim\limits_{a\to\infty}\int_1^\infty \dfrac{dx}{x} = \lim\limits_{a\to\infty}\ln a - \ln 1 =$ *infinity*, so III. diverges.

26. **E** Area under polar curve is found by using: $A = \int_a^b \dfrac{1}{2} r^2\, d\theta$. This curve repeats after 2π, thus the region is bound by 0 and 2π. The area is then found from: $A = \int_0^{2\pi} \dfrac{1}{2}(5 + 2\cos\theta)^2 d\theta$. Solve this and the area is found to be 27π.

27. **A** Solve the differential equation by separating the variables and integrating. The equation would then become: $\int y^2\, dy = \int \cos x\, dx$. From here, $y^3 = 3\sin x + C$. Next, use the initial condition (0, –1) to get the exact equation: $y^3 = 3\sin x - 1$.

28. **D** A series converges to L when $\lim\limits_{n\to\infty} a_n = L$. There are many tests that can be used to test whether a series will converge. The test depends on the type of series. So, check each series in this problem one at a time and POE. I. is a geometric series. Geometric series converge if $|r| < 1$ and the general form of the series is $\sum\limits_{n=1}^{\infty} ar^{n-1}$. The r in I. is $\dfrac{1}{3}$, so the series converges. II. can be tested using the ratio test which states that a series in the form $\sum a_n$ converges if $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} < 1$. Then II. can be tested as follows, $\lim\limits_{n\to\infty}\dfrac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \dfrac{1}{3}$. Since $\dfrac{1}{3}$ is less than 1, the series converges. For III., the integral test can be used. The integral test states $\sum a_n$ and $\int_1^\infty f(x)\, dx$ either both converge or both diverge, for $a_n = f(n)$. Therefore, evaluate the improper integral $\int_1^\infty \dfrac{6x^2}{x^3+1}dx$. This integral diverges (it equals infinity), so the series diverges.

29. **B** The balloon losing air at a rate of -24π in^3/sec means the volume is shrinking at that rate, but it does not directly relate to surface area. Therefore, we must find an intermediate rate to relate the two formulas, $V = \dfrac{4}{3}\pi r^3$ and $A = 4\pi r^2$. The common value is r, so the common rate will be $\dfrac{dr}{dt}$. To solve for $\dfrac{dr}{dt}$, differentiate the formula for V with respect to t: $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. Insert the values for $\dfrac{dV}{dt}$ and r and solve for $\dfrac{dr}{dt}$. Thus, $\dfrac{dr}{dt} = -\dfrac{3}{2}$ in. Next, differentiate the formula for A with respect to t: $\dfrac{dA}{dt} = 8\pi r\dfrac{dr}{dt}$. Insert the values for $\dfrac{dr}{dt}$ and r to solve for $\dfrac{dA}{dt}$. Then, $\dfrac{dA}{dt} = -24\pi$ in^2/sec.

30. **A** When you insert 0 for x, the limit is $\dfrac{0}{0}$, which is indeterminate. Use L'Hôpital's Rule to evaluate the limit: $\lim\limits_{x\to 0}\dfrac{3(3+2x)^{\frac{1}{2}} - 7}{4x}$. Since this limit is also indeterminate, use L'Hôpital's Rule again: $\lim\limits_{x\to 0}\dfrac{3(3+2x)^{-\frac{1}{2}}}{4}$. This limit exists and equals $\dfrac{\sqrt{3}}{4}$.

31. **C** The horizontal and vertical components of the velocity of a curve are found parametrically as the derivatives with respect to time of the x and y components of the motion, respectively. For this curve, $\dfrac{dx}{dt} = 5t$ and $\dfrac{dy}{dt} = 6t^2 - 2t + 1$. When those two expressions are set equal to each other and solved for t, $t = 1$ and $t = \dfrac{1}{6}$. Since $t = 1$ is not an answer choice, C is the correct answer.

32. **D** Use implicit differentiation to find $\frac{dy}{dx}$:

$$2x + 6xy + 3x^2\frac{dy}{dx} + 3y^2 + 6xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

Do not rearrange the terms to isolate $\frac{dy}{dx}$. Instead, plug in (3,2) immediately, solve for the derivative, and simplify:

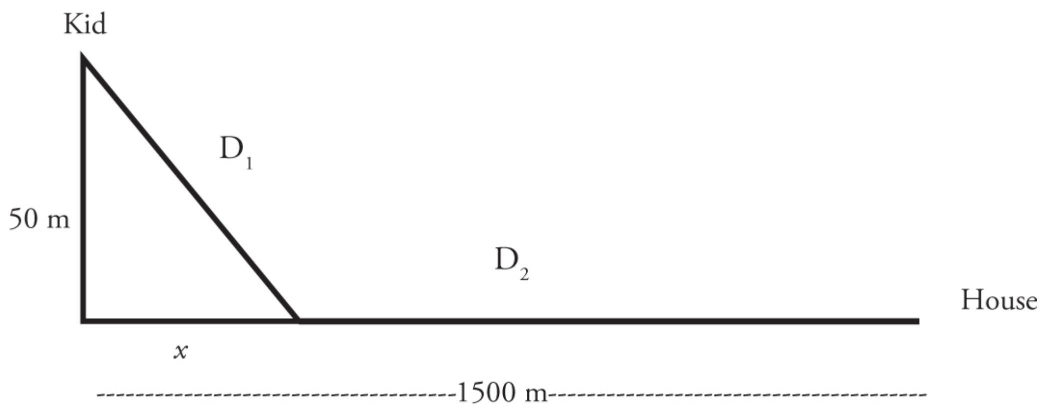
$$2(3) + 6(3)(2) + 3(3)^2\frac{dy}{dx} + 3(2)^2 + 6(3)(2)\frac{dy}{dx} + 3(2)^2\frac{dy}{dx} = 0$$

$$6 + 36 + 27\frac{dy}{dx} + 12 + 36\frac{dy}{dx} + 12\frac{dy}{dx} = 0$$

$$54 + 75\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{54}{75} = -\frac{18}{25}$$

33. **E**



From the diagram, it is clear $D_1 = \sqrt{50^2 + x^2} = \sqrt{2500 + x^2}$ and $D_2 = 1500 - x$. Since distance = rate • time and $r_1 = 3$ and $r_2 = 5$, the times for each leg of the trip are $t_1 = \frac{\sqrt{2500 + x^2}}{3}$ and $t^2 = \frac{1500 - x}{5}$. Therefore, the total time to ride home is $T = \frac{\sqrt{2500 + x^2}}{3} - \frac{x}{5} + 300$. To minimize the time, the derivative of T is taken and set equal to zero. $\frac{dT}{dx} = \frac{1}{3} \bullet \frac{1}{2} \bullet 2x \bullet \left(2500 + x^2\right)^{-\frac{1}{2}} - \frac{1}{5} = 0$ and $x = \pm\frac{75}{2}$. The negative value can be ignored. Because the derivative is ugly, it will take too much time to check the second derivative. (Don't make skipping this a habit, though!) Therefore, the time is minimized when $x = \frac{75}{2}$ m. The kid should cross at 1462.5 m from his house.

34. **B** The profit is maximized at the values that make the derivative of the profit equation equal to zero or at the end points of the range. The derivative of the profit equation is $\frac{dP}{dx} = 6x^2 - 210x - 1500 = 0$. This equation is true when $x = 10$ or 25. Then, those values and the endpoints, $x = 0$ and $x = 50$ are used to solve for P. The resulting points are (0, –1200), (10, 5300), (25, 1925), and (50, 61300). Since the profit equation is maximized at $x = 50$, and the profit is in millions of dollars, the final result is \$61.3 billion.

35. **B** Use the mean value theorem for integrals, $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$. Thus, for this problem, $5 = \frac{1}{k-0}\int_0^k x^4\, dx$. Using the Fundamental Theorem of Calculus and solving the equation for k, $k = \pm\sqrt{5}$.

36. **A** Recall, $\int a^u\, du = \frac{a^u}{\ln a} + C$. In this problem, $u = 3x^2 + 2$ and $du = 6\, x\, dx$.

Thus, $\frac{1}{2}\int 7^u\,du = \frac{7^u}{2\ln 7}+C = \frac{7^{3x^2+2}}{\ln 49}+C = \frac{49 \bullet 343^{x^2}}{\ln 49} + C$

37. **B** The area of a rectangle is $A = lw$. In this case, the width is $2x$ and the length $\cos x + 1$, so $A = 2x(\cos x + 1)$. To minimize the area between the rectangle and the graph, the area of the of the rectangle will be maximized. To maximize the area of the rectangle, you must take the derivative of the area and set it equal to zero: $\frac{dA}{dx} = 2\cos x - 2x\sin x + 2 = 0$. When you solve this with your calculator, there will be four critical points: $-\pi$, -1.30654, 1.30654, π. Plug the points into the formula to determine the area at each point. At $-\pi$ and π, the area is 0. At -1.30654 and 1.30654, the area is 3.29559. (You can confirm this is the maximum by taking the second derivative and checking that it is negative at 1.30654.) Next, determine the area under the curve by using a definite integral. $\int_{-\pi}^{\pi} \cos x + 1dx = 2\pi$. To determine the area between the rectangle and the curve, subtract the area of the rectangle from the area under the curve: $A_{\text{between}} = 2\pi - 3.29559 = 2.98759$.

38. **D** The current state of the integral makes it appear very difficult to solve. However, notice that there is an x^2 term and no square root. So, think inverse tangent! First, complete the square: $x^2 + 6x + 10 = (x + 3)^2 + 1$. Thus, you can rewrite the integral as $\int \frac{dx}{(x+3)^2+1}$. Now, $u = x + 3$ and $du = dx$. Use the pattern $\int \frac{dx}{1+u^2} = \tan^{-1} u + C$. Therefore, $\int \frac{dx}{(x+3)^2+1} = \tan^{-1}(x + 3) + C$.

39. **E** In order to find the distance the particle travels, set the first derivative equal to zero to determine, when, if at all, it changes direction over the time interval: $x'(t) = 4t - 3 = 0$, hence $t = \frac{3}{4}$. Since the particle changes direction at $t = \frac{3}{4}$, we must find the position of the particle at $t = 0$, $t = \frac{3}{4}$, and $t = 4$. $x(0) = 1$, $x\left(\frac{3}{4}\right) = -\frac{1}{8}$, and $x(4) = 21$. The total distance the particle travels is found by $d = \left|x\left(\frac{3}{4}\right)-x(0)\right| + \left|x(4)-x\left(\frac{3}{4}\right)\right| = \left|-\frac{1}{8}-1\right| + \left|21+\frac{1}{8}\right| = \frac{89}{4}$.

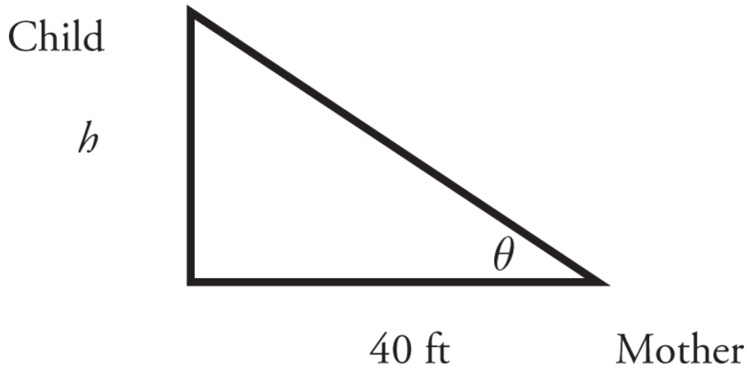
40. **E** Given a table of values, a left-hand Riemann sum can be calculated by multiplying the size of the intervals (i.e., the difference in x-values), by the left endpoint y-values and summing them all up.

41. **C** Recall $L = \int_a^b \sqrt{1++\left(\frac{dy}{dx}\right)^2}\,dx$. In this case, $\frac{dy}{dx} = x(x^2 - 2)^{\frac{1}{2}}$ and $L = \int_0^4 \sqrt{1++\left(x\left(x^2-2\right)^{\frac{1}{2}}\right)^2}\,dx = \frac{64}{3}-4$.

42. **E** From the Second Fundamental Theorem of Calculus, $\frac{d}{dx}\int_1^x (t - t^4)\,dt = x - x^4$.

43. **D** This is a related rates problem. Since the volume of a cylinder is given by $V = \pi r^2 h$, begin by taking the derivative of both sides of the equation with respect to t. Because the radius is constant, treat it as a constant when taking the derivative: $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$. Now, plug in the given values for $\frac{dV}{dt}$ and r: $96\pi = \pi 4^2 \frac{dh}{dt}$. Solve for $\frac{dh}{dt} = 6$.

44. **B** Draw a picture of the situation:



The question is asking about θ in this right triangle and we are given the measurements of the two legs, so set up your equation using tan: $\tan\theta = \frac{h}{40}$. As this is a related rates problem, take the first derivative with respect to *t*: $\sec^2\theta\frac{d\theta}{dt} = \frac{1}{40}\frac{dh}{dt}$. Notice the rate of the child rising, $\frac{dh}{dt}$, was given in ft/min while the question asks for the solution in rad/sec. Convert $\frac{dh}{dt}$ into ft/sec: $\frac{dh}{dt} = \frac{1}{6}$ ft/sec. Also, note that a height, *h,* was given in the problem. Plug that into the original equation for tan to solve for θ or directly solve for $\sec^2\theta$: $\tan\theta = \frac{30}{40} = \frac{3}{4}$. Recall, $1 + \tan^2\theta = \sec^2\theta$, so plug in to solve for $\sec^2\theta$: $\sec^2\theta = \frac{25}{16}$. Now, plug in all these values into the derivative of tan and solve for $\frac{d\theta}{dt}$: $\frac{25}{16}\frac{d\theta}{dt} = \frac{1}{40}\left(\frac{1}{6}\right)$ or $\frac{d\theta}{dt} = \frac{1}{375}$ rad/sec.

45. **C** Use the quotient rule and recall the derivative of the natural log and *e*.

$$\frac{dy}{dx} = \frac{\left(e^{2x}\right)\left(\frac{6x^2}{2x^3}\right) - 2e^{2x}\ln\left(2x^3\right)}{e^{4x}} = \frac{3}{xe^{2x}} - \frac{2\ln\left(2x^3\right)}{e^{2x}}.$$

Section II

1. Let *y* be the function satisfying $f'(x) = -x(1 + f(x))$ and $f(0) = 5$.

(a) Use Euler's method to approximate $f(1)$ with a step size of 0.25.

(a) There are two equations you need to know for Euler's method: 1. $x_n = x_{n-1} + h$ and 2. $y_n = y_{n-1} + hy'_{n-1}$. In this case, there will be four steps ($n = 4$) and $h = 0.25$. From the given information, $y' = -x(1 + y)$, so $y'_0 = 0$. It is also given that $x_0 = 0$ and $y_0 = 5$. Then, $x_1 = 0.25$, $y_1 = 5$, and $y'_1 = 1.5$. Next, $x_2 = 0.5$, $y_2 = 4.625$, and $y'_2 = -2.8125$. Once more, $x_3 = 0.75$, $y_3 = 3.92188$, and $y'_3 = -3.69141$. Finally, $x_4 = 1$ and $y_4 = 2.999$.

(b) Find an exact solution for $f(x)$ when $x = 1$.

(b) Solve the differential equation: $\frac{dy}{dx} = -x(1 + y)$. Then, $y = Ce^{-\frac{x^2}{2}} - 1$. With the initial condition that $f(0) = 5$, $f(x) = 6e^{-\frac{x^2}{2}} - 1$. Therefore, $f(1) = 6e^{-\frac{1}{2}} - 1 \approx 2.639$.

(c) Evaluate $\int_0^\infty -x(1 + f(x))dx$.

(c) First, remember the Fundamental Theorem of Calculus: $\int_a^b f'(x)dx = f(b) - f(a)$. From part (b), $f(x) = 6e^{-\frac{x^2}{2}} - 1$. Notice that $\int_0^\infty -x(1 + f(x))dx$ is an improper integral. So, evaluate $\lim_{a\to\infty} f(a)$ and $f(0)$ and find the difference. $\lim_{a\to\infty} f(a) = -1$ and $f(0) = 5$. Therefore, $\int_0^\infty -x(1 + f(x))dx = -6$.

2. Let R be the region enclosed by the graphs of $y = x^2 - x - 6$ and $y = x^3 - 2x^2 - 5x + 6$ and the lines $x = -2$ and $x = 2$.

(a) Find the area of R.

(a) *Area* $= \int_a^b [f(x) - g(x)]dx$. We are told the region is bound by $x = -2$ and $x = 2$. Further, $y = x^3 - 2x^2 - 5x + 6$ is always more positive or "on top" over the entire region, so $f(x) = x^3 - 2x^2 - 5x + 6$ and $g(x) = x^2 - x - 6$. Now, the integral can be set up: $A = \int_{-2}^{2} (x^3 - 2x^2 - 5x + 6) - (x^2 - x - 6)dx$. When solved, the area is found to be 32 units squared.

(b) The horizontal line $y = 0$ splits R into two parts. Find the area of the part of R above the horizontal line.

(b) The line $y = 0$ divides R so that the area above R is the region enclosed by $y = 0$ and $y = x^3 - 2x^2 - 5x + 6$. Now, the limits of integration are when these two curves intersect. These curves intersect at $x = -2$, $x = 1$, and $x = 3$. However, R is bound between $x = -2$ and $x = 2$, so the limits of integration are $x = -2$ and $x = 1$. The same formula for area can be used from part (a), but the specific values input would be the ones described here. Thus, $A = \int_{-2}^{1} (x^3 - 2x^2 - 5x + 6 - 0)dx$. Thus, the area of this portion of $R = \frac{63}{4}$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is an equilateral triangle. Find the volume of this solid.

(c) The volume of the solid is found by integrating the area of the cross-section, so $V = \int_a^b A(x)dx$. In this case, since the cross-section is an equilateral triangle, $A = (side)^3\frac{\sqrt{3}}{4}$, where the side length is the length between the curves, i.e., $f(x) - g(x)$. From part (a), we know and $g(x) = x^2 - x - 6$ and $g(x) = x^2 - x - 6$ and the limits of integration are $x = -2$ and $x = 2$. The equation for the volume is then $V = \int_{-2}^{2} \frac{\sqrt{3}}{4} [(x^3 - 2x^2 - 5x + 6) - (x^2 - x - 6)]dx = \frac{8576\sqrt{3}}{105} = 141.467$ units cubed.

(d) What is the volume of the solid generated by the region R, when it is revolved about the line $x = -3$.

(d) Since you are not told which method to use to find the volume you must decide. You can use the rule of thumb that it is typically (but not always) better to use cylindrical shells, if the region is bound by more than two curves (including an axis) or if one or more curves are given as $y =$ and the others are given as $x =$. Both conditions are satisfied in this problem, so cylindrical shells is probably best. The general formula for cylindrical shells is $2\pi\int_a^b x[f(x) - g(x)]dx$. We know from part (a) the limits of integration are $x = -2$ and $x = 2$ and we have established which curve is $f(x)$ and which is $g(x)$ in part (a). The only thing left to establish is the radius of the cylinder from the axis of rotation. Since there is a shift in the axis to $x = -3$, the radius is $x + 3$. Thus, the final integral is: $V = 2\pi\int_{-2}^{2} (x + 3)[(x^3 - 2x^2 - 5x + 6) - (x^2 - x - 6)]dx = \frac{6046\pi}{35} = 542.688$ units cubed.

3. The derivative of a function f is $f'(x) = (2x + 6)e^{-x}$ and $f(2) = 15$.

(a) The function has a critical point at $x = -3$. Is there a relative maximum, minimum, or neither at this point on f? Justify your response.

(a) You have two options here: 1) Using the first derivative test: Since $x = -3$ is a critical point, a local minimum will exist when $f'(x) < 0$ for $x < -3$ and $f'(x) > 0$ for $x > -3$. In this case, those two criteria are satisfied, so $x = -3$ is a relative minimum. 2) You can use the second derivative test: Take the derivative of $f'(x)$. $f''(x) = (-2x - 4)e^{-x}$. Plug $x = -3$ into $f''(x)$. From the second derivative test, if $f''(c) < 0$, then c is a relative maximum, but if $f''(c) > 0$, then c is a relative minimum. In this case, $f''(-3) = 2e^{-3}$ which is greater than 0, so $x = -3$ is a relative minimum of f.

(b) On what interval, if any, is the graph of f both increasing and concave down? Explain your reasoning.

(b) First, use the second derivative from part (a). Find the points of inflection by setting $f''(x) = 0$. There is a point of inflection at $x = -2$. Now, test the behavior of $f''(x)$ around $x = -2$. When $x < -2$, $f''(x) > 0$, which means the curve is “concave up.” When $x > -2$, $(x) < 0$, which means the curve is “concave down.” To determine when the curve is decreasing, the first derivative must be tested. We already know from the question stem to part (a) that there is a critical point at $x = -3$. So, test the behavior of $f'(x)$ around $x = -3$. For $x < -3$, $(x) < 0$, and, thus, falling or decreasing. For $x > -3$, $f'(x) > 0$ and, thus, rising or increasing. When the behaviors of the first and second derivatives of f are combined, $f(x)$ is increasing and concave down over the interval $x > -2$.

(c) Find the value of a $f(5)$.

(c) In order to find $f(5)$, you can integrate to find f and then use the initial condition $f(2) = 15$ to find the exact solution. Once the exact solution (the exact equation) is found, then f can be evaluated at $x = 5$. A more direct path would be to integrate (x) over the range from $x = 2$ to $x = 5$. The value found can then be added to $f(2)$. So the solution would be $f(5) = f(2) + \int_2^5 (2x + 6)e^{-x}\, dx = 15 - 18e^{-5} + 12e^{-2}$.

4. Consider the equation $x^3 - 2x^2y + 3xy^2 - 4y^3 = 10$.

(a) Write an equation for the slope of the curve at any point.

(a) Find the first derivative of this equation using implicit differentiation:

$$3x^2 - \left(2x^2\frac{dy}{dx} + 4xy\right) + \left(6xy\frac{dy}{dx} + 3y^2\right) - 12y^2\frac{dy}{dx} = 0$$

$$3x^2 - 4xy + 3y^2 = \frac{dy}{dx}\left(2x^2 - 6xy + 12y^2\right)$$

$$\frac{dy}{dx} = \frac{3x^2 - 4xy + 3y^2}{2x^2 - 6xy + 12y^2}$$

(b) Find the equation of the normal line to the curve at the point $x = 1$.

(b) At $x = 1$, $y = -1$. Plug these values into the equation for $\frac{dy}{dx}$ from part (a). Then, the slope of the tangent line is 1. The slope of the normal line is the opposite reciprocal, which is –1. Using point-slope formula, the equation of the normal is $y + 1 = x - 1$ or $y = x - 2$.

(c) Find $\frac{d^2y}{dx^2}$ at $x = 1$.

(c) The second derivative follows directly from the first in part (a):

$$\frac{d^2y}{dx^2} = \frac{(2x^2 - 6xy + 12y^2)\left(6x - 4x\frac{dy}{dx} - 4y + 6y\frac{dy}{dx}\right) - (3x^2 - 4xy + 3y^2)\left(4x - 6x\frac{dy}{dx} - 6y + 24y\frac{dy}{dx}\right)}{(2x^2 - 6xy + 12y^2)^2}$$

Rather than trying to simplify the equation, plug in the value for $\frac{dy}{dx}$ from part (b) and $x = 1$ and $y = -1$. When simplified using those values, $\frac{d^2y}{dx^2} = \frac{1}{2}$.

5. Given that $f(x) = \sin x$:

(a) Find the 6^{th} degree Maclaurin series.

(a) In order to determine the 6^{th} degree Maclaurin series, use the general formula: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x)^k = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$. So, find $f(x)$ and the first 5 derivatives of $f(x)$:

$f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, and $f^{(5)}(x) = \cos x$

Then, evaluate each of those equations at $x = 0$:

$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f^{(4)}(0) = 0$, and $f^{(5)}(0) = 1$. Finally, insert the values into the formula for a Maclaurin series and simplify: $\sin x = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} = x - \frac{x^3}{3!} + \frac{x^5}{5!}$.

(b) Use the polynomial to estimate sin 0.2.

(b) From part (a), $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$. To approximate sin 0.2, plug 0.2 in for x in the equation and simplify: $\sin 0.2 \approx 0.2 - \frac{0.2^3}{3!} + \frac{0.2^5}{5!} \approx 0.201336$.

(c) Estimate the remainder of the approximation.

(c) The formula for the Lagrange remainder is $R_n(x, a) \le f^{(n+1)}(c)\frac{(x-a)^{n+1}}{(n+1)!}$. Use the series you found in part (a) to generalize the series $\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$. From here, $R_{2n}(x, 0) \le \frac{x^{2n+1}}{(2n+1)!}\sin(c + \frac{(2n+1)\pi}{2})$. (in this case, it is R_{2n} not R_n because every other term is 0.) Recall, $|\sin x| \le 1$, so $R_{2n+1}(x,0) \le \frac{|x|^{2n+1}}{(2n+1)!}$. Therefore, $R_7(x, 0) \le \frac{x^7}{(7)!}$. (In general, a good approximation of the remainder/error bound of an nth degree Taylor polynomial is the next nonzero term in a decreasing series.) For the approximation in part (b), plug 0.2 for x into $\frac{x^7}{(7)!}$. Therefore, the remainder is $\frac{(0.2)^7}{(7)!} \approx 2.540 \times 10^{-9}$.

6. Two particles travel in the xy-plane for time $t \ge 0$. The position of particle A is given by $x = 2t - 3$ and $y = (2t + 1)^2$ and the position of particle B is given by $x = t - 1$ and $y = t + 23$.

(a) Find the velocity vector for each particle at $t = 3$.

(a) The velocity vector of each particle is found by taking the derivative of each particle's motion, so the vector is $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$. For particle A, $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 8t - 4$. For particle B, $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 1$. Evaluate each of these derivatives at $t = 3$, so the final velocity vectors are $A = (2,20)$ and $B = (1,1)$.

(b) Set up, but do not evaluate, an integral expression for the distance traveled by particle A from $t = 3$ to $t = 5$.

(b) The formula for the distance travelled parametrically, or the length of the curve, is $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. Since the velocity vector does not equal zero at any time over the interval from $t = 3$ to $t = 5$, the particle does not change direction and we can plug in the formulas for the x and y components of the velocity from part (a): $L = \int_2^5 \sqrt{2^2 + (8t-4)^2}\, dt$, which is simplified to $L = \int_2^5 \sqrt{64t^2 - 64t + 20}\, dt$.

(c) At what time do the two particles collide? Justify your answer.

(c) The particles will collide when the x and y coordinates of their position functions are equal to each other. Therefore, set the position functions equal and solve for t. Beginning with the x-components: $2t - 3 = t - 1$, so $t = 2$. Plug $t = 2$ into the y-components and confirm they are equal. For particle A: $y = (2 \cdot 2 + 1)^2 = 25$. For particle B: $y = 2 + 23 = 25$. Both the x and y components are equal at $t = 2$, so that is when they collide.